

# On the rate analysis of inexact augmented Lagrangian schemes for convex optimization problems with misspecified constraints

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**Abstract**—We consider a misspecified optimization problem that requires minimizing of a convex function  $f(x; \theta^*)$  in  $x$  over a constraint set represented by  $h(x; \theta^*) \leq 0$  where  $\theta^*$  is an unknown (or misspecified) vector of parameters. Suppose  $\theta^*$  is learnt by a distinct process that generates a sequence of estimators  $\theta_k$ , each of which is an increasingly accurate approximation of  $\theta^*$ . We develop a first-order augmented Lagrangian scheme for computing an optimal solution  $x^*$  while simultaneously learning  $\theta^*$ .

## I. INTRODUCTION

Consider an optimization problem in  $n$ -dimensional space defined as follows:

$$\mathcal{X}^*(\theta^*) := \operatorname{argmin}_{x \in X \cap \mathcal{H}(\theta^*)} f(x; \theta^*), \quad (\mathcal{C}(\theta^*))$$

where  $\theta^* \in \mathbb{R}^d$  denotes the parametrization of the objective and constraints. While traditionally, optimization research has considered settings where  $\theta^*$  is available a priori, two related problems of interest have considered regimes where either the parameter is unavailable (*robust* optimization) or when it is uncertain (*stochastic* optimization):

**Robust approaches** [1]. For instance, when  $\theta^*$  is unavailable, but one has access to an associated uncertainty set  $\mathcal{T}$ , then in robust optimization, the worst-case value of the objective is minimized:

$$\min_{x \in X} \max_{\theta \in \mathcal{T}} f(x; \theta). \quad (\text{Robust Optimization})$$

**Stochastic approaches** [2]. An alternative approach considers an uncertain regime where  $\theta : \Omega \rightarrow \mathbb{R}^d$  is an  $d$ -dimensional random vector defined on a suitable probability space. The resulting stochastic optimization schemes consider the minimization of an expectation:

$$\min_{x \in X} \mathbb{E}[f(x; \theta)]. \quad (\text{Stochastic Optimization})$$

In this paper, we consider a different approach in which the parameter vector  $\theta$  has a nominal or true value  $\theta^*$  obtainable by solving a suitably defined learning problem:

$$\min_{\theta \in \Theta} \ell(\theta). \quad (\mathcal{E})$$

Instances of such problems routinely arise when  $\theta^*$  is idiosyncratic to the problem and may be learnt by the aggregation of data; examples include the following: the learning of covariance matrices associated with a collection of stocks,

efficiency parameters associated with machines on a supply line, and demand parameters associated with a supply chain. A natural approach in this case is to first estimate  $\theta^*$  with high accuracy and then to solve the parametrized problem. Yet, in many instances, this *sequential* approach cannot be adopted for several reasons: (i) observations unavailable a priori and appear in a streaming fashion; (ii) the learning problem can be large, precluding a highly accurate a priori resolution; (iii) unless the learning problem can be solved *exactly* in finite time, any sequential scheme may provide approximate solutions, at best.

Accordingly, we consider the development of schemes that generate sequences  $\{x_k\}, \{\theta_k\}$  such that

$$\|\theta_k - \theta^*\| \rightarrow 0, \quad d_{\mathcal{X}^*(\theta^*)}(x_k) \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

where  $\theta^*$  is the unique solution of  $(\mathcal{E})$  and for a given closed convex set  $\mathcal{X}$ ,  $d_{\mathcal{X}}(x) \triangleq \min_{s \in \mathcal{X}} \|x - s\|$  denotes the distance function to  $\mathcal{X}$ . This framework originates from prior work on stochastic optimization/variational inequality problems [3] and stochastic Nash games [4]. In recent work, the rate statements derived in [3] are refined to the deterministic regime [5]. In [6], misspecification in the constraints is addressed in a convex regime via variational inequality approaches; in sharp contrast, in this paper, we develop a misspecified analog of the augmented Lagrangian scheme for misspecified convex problems in which both the objective and the constraints are misspecified. Augmented Lagrangian schemes are rooted in the seminal papers by Hestenes [7] and Powell [8], and their relation to proximal-point methods was established by Rockafellar [9], [10]. Recently, there has been a renewed examination of such techniques in convex regimes, with an emphasis on deriving convergence rates [11]–[13].

In this paper, we develop an analog of the traditional augmented Lagrangian scheme in which the subproblems are solved with increasing exactness. Our contributions include rate statements for the dual suboptimality, the primal infeasibility, and the primal suboptimality in this misspecified regime. Throughout, our focus will be on the problem  $(\mathcal{C}(\theta^*))$  when  $\mathcal{H}(\theta^*) \triangleq \{x : h(x; \theta^*) \leq 0\}$ , i.e.,

$$\begin{aligned} \min_x \quad & f(x; \theta) \\ \text{subject to} \quad & h(x; \theta) \leq 0, \quad x \in X, \end{aligned} \quad (\mathcal{C}(\theta))$$

where  $f : \mathbb{R}^n \times \Theta \rightarrow \mathbb{R} \cup \{+\infty\}$ ,  $h : \mathbb{R}^n \times \Theta \rightarrow \mathbb{R}^m$  and  $\theta \in \Theta \subseteq \mathbb{R}^d$  denotes an estimate for the misspecified parameter  $\theta^*$ . Throughout, we assume that  $\mathcal{C}(\theta^*)$  has a finite optimal value, given by  $f^*$ , the corresponding Lagrangian dual problem has a solution, denoted by  $\lambda^*$ , and there is *no*

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duality gap. The remainder of the paper comprises of three sections. We provide preliminaries in Section II, the main rate statements in Section III, and conclude in Section IV.

## II. PRELIMINARIES

The problem  $\mathcal{C}(\theta)$  is equivalent to

$$\min \{f(x; \theta) : h(x; \theta) + z = 0, \quad x \in X, \quad z \in \mathbb{R}_+^m\}. \quad (1)$$

Let  $\lambda \in \mathbb{R}^m$  denote the vector dual variables corresponding to the equality constraints in (1). For any given  $\rho > 0$ , define the augmented Lagrangian function for (1),  $\mathcal{L}_\rho(x, \lambda; \theta)$ , such that  $\text{dom } \mathcal{L}_\rho = X \times \mathbb{R}^m \times \Theta$  and

$$\mathcal{L}_\rho(x, \lambda; \theta) \triangleq \min_{z \in \mathbb{R}_+^m} \left[ f(x; \theta) + \lambda^\top (h(x; \theta) + z) + \frac{\rho}{2} \|h(x; \theta) + z\|^2 \right].$$

Through a rearrangement of terms, it can be shown that

$$\begin{aligned} \mathcal{L}_\rho(x, \lambda; \theta) &= f(x; \theta) + \frac{\rho}{2} \min_{z \in \mathbb{R}_+^m} \left\| h(x; \theta) + z + \frac{\lambda}{\rho} \right\|^2 - \frac{\|\lambda\|^2}{2\rho} \\ &= f(x; \theta) + \frac{\rho}{2} d_{\mathbb{R}_+^m}^2 \left( \frac{\lambda}{\rho} + h(x; \theta) \right) - \frac{\|\lambda\|^2}{2\rho}, \end{aligned} \quad (2)$$

where  $d_{\mathcal{X}}(x) \triangleq \min_{s \in \mathcal{X}} \|x - s\|$ , and  $d_{\mathcal{X}}^2(x) \triangleq (d_{\mathcal{X}}(x))^2$ . For  $\rho = 0$ , let  $\mathcal{L}_0(x, \lambda; \theta)$  denote the Lagrangian function:

$$\mathcal{L}_0(x, \lambda; \theta) \triangleq \begin{cases} f(x; \theta) + \lambda^\top h(x; \theta), & \text{if } \lambda \in \mathbb{R}_+^m \\ -\infty, & \text{otherwise.} \end{cases}$$

When  $\rho \geq 0$ , the dual problem of  $\mathcal{C}(\theta)$  is defined as

$$\max_{\lambda \in \mathbb{R}_+^m} \left\{ g_\rho(\lambda; \theta) \triangleq \inf_{x \in X} \mathcal{L}_\rho(x, \lambda; \theta) \right\}. \quad (D_\rho)$$

Throughout, we assume the following:

*Assumption 1:*

- (i) The functions  $f(x, \theta)$  and  $h_i(x, \theta)$  are convex in  $x \in X$  for all  $\theta \in \Theta$  for  $i = 1, \dots, m$  and  $X \subseteq \mathbb{R}^n$  and  $\Theta$  are convex compact sets.
- (ii) The function  $f(x; \theta)$  is Lipschitz continuous in  $\theta$  over  $\Theta$  for all  $x \in X$  with constant  $L_f$ ; i.e. for all  $x \in X$ ,  $\|f(x; \theta_1) - f(x; \theta_2)\| \leq L_f \|\theta_1 - \theta_2\|$  for all  $\theta_1, \theta_2 \in \Theta$ .
- (iii)  $h(x; \theta)$  is an affine map in  $x$  for every  $\theta \in \Theta$ , i.e.,  $h(x; \theta) = A(\theta)x + b(\theta)$  for some  $A(\theta) \in \mathbb{R}^{m \times n}$  and  $b(\theta) \in \mathbb{R}^m$ . Suppose  $A(\theta)$  and  $b(\theta)$  are Lipschitz continuous in  $\theta$ . Hence,  $h(x; \theta)$  is Lipschitz continuous in  $\theta$  with constant  $L_h$  uniformly for all  $x \in X$ . Clearly,  $h(B(0, 1); \theta) \subseteq B(b(\theta), \sigma_{\max}(A(\theta)))$  for all  $\theta \in \Theta$ , where  $B(\bar{y}, r) := \{y : \|y - \bar{y}\| \leq r\}$ . Hence, there is a constant  $\sigma_h$  s.t.  $h(B(0, 1); \theta) \subseteq \sigma_h B(0, 1)$  for all  $\theta \in \Theta$ , since  $\sigma_{\max}(A(\theta))$  is continuous in  $\theta$  and  $\Theta$  is compact.
- (iv)  $X^*(\lambda; \theta)$  is pseudo-Lipschitz in  $\theta$  uniformly in  $\lambda$  with constant  $\kappa_X$ , where  $X^*(\lambda; \theta) = \arg \min_{x \in X} \mathcal{L}_0(x, \lambda; \theta)$ , i.e., for any  $\theta_1, \theta_2 \in \Theta$ ,  $X^*(\lambda; \theta_1) \subseteq X^*(\lambda; \theta_2) + \kappa_X B(0, 1)$  for all  $\lambda \geq 0$ .

Rather than focusing on the nature of the algorithm employed for resolving the learning problem, instead we impose a requirement that the adopted scheme produces a sequence that converges to the optimal solution at a non-asymptotic linear rate.

*Assumption 2:* There exists a learning scheme that generates a sequence  $\{\theta_k\}$  such that  $\theta_k \rightarrow \theta^*$  at a linear rate as  $k \rightarrow \infty$ , i.e., there exists a constant  $q_\ell \in (0, 1)$  such that for all  $k \geq 0$  and  $\theta_0 \in \Theta$ , one has  $\|\theta_k - \theta^*\| \leq q_\ell^k \|\theta_0 - \theta^*\|$ . In addition, at iteration  $k$  of the optimization problem  $\mathcal{C}$ , only  $\theta_1, \dots, \theta_k$  are revealed.

Lemma 1, pertaining to various properties of the gradient of the dual function  $\nabla_\lambda g_\rho$ , will be used in our analysis. The proof of Lemma 1 can be found in [10] and is omitted here.

*Lemma 1:* Suppose Assumption 1 holds.

- (i) For any  $\rho > 0$  and  $\theta \in \Theta$ , the dual function  $g_\rho(\lambda; \theta)$  is everywhere finite, continuously differentiable concave function over  $\mathbb{R}^m$ ; more precisely,  $g_\rho(\lambda; \theta) = \max_{w \in \mathbb{R}^m} \{g_0(w; \theta) - \frac{1}{2\rho} \|w - \lambda\|^2\}$ , i.e.,  $g_\rho(\cdot, \theta)$  is the Moreau regularization of  $g_0(\cdot, \theta)$  for all  $\theta \in \Theta$ . Therefore,  $\nabla_\lambda g_\rho(\lambda; \theta)$  is Lipschitz continuous in  $\lambda$  with constant  $\frac{1}{\rho}$ .
- (ii) For any given  $\lambda \in \mathbb{R}_+^m$  and  $\theta \in \Theta$ ,  $\nabla_\lambda g_\rho$  can be computed as  $\nabla_\lambda g_\rho(\lambda; \theta) = \nabla_\lambda \mathcal{L}_\rho(x^*(\lambda), \lambda; \theta)$ , where  $x^*(\lambda) \in \arg \min_{x \in X} \mathcal{L}_\rho(x, \lambda; \theta)$ .
- (iii) Given  $\lambda \in \mathbb{R}_+^m$  and  $\theta \in \Theta$ , suppose  $\tilde{x}(\lambda)$  is an inexact solution to  $\min_{x \in X} \mathcal{L}_\rho(x, \lambda; \theta)$  with accuracy  $\alpha$ , i.e.,  $\tilde{x}(\lambda) \in X$  satisfies  $\mathcal{L}_\rho(\tilde{x}(\lambda), \lambda; \theta) \leq g_\rho(\lambda; \theta) + \alpha$ , then

$$\|\nabla_\lambda \mathcal{L}_\rho(\tilde{x}(\lambda), \lambda; \theta) - \nabla_\lambda g_\rho(\lambda; \theta)\|^2 \leq 2\alpha/\rho.$$

Next, we examine the continuity of  $\nabla_\lambda g_\rho(\lambda; \theta)$  in  $\theta \in \Theta$ .

*Lemma 2 (Lipschitz continuity of  $\nabla_\lambda g_\rho$  in  $\theta \in \Theta$ ):*

Suppose Assumption 1 holds. Then, we have that  $\nabla_\lambda g_\rho(\lambda; \theta)$  is Lipschitz continuous in  $\theta$  over  $\Theta$  uniformly in  $\lambda \in \mathbb{R}^m$  with constant  $\kappa_h + \kappa_X \sigma_h$ .

*Proof:* Due to limited space, we omit the proof. For details, see Proposition 2.4 in the extended version of this paper [14].

**Remark:** We now comment on the conditions under which  $X^*(\lambda; \theta)$  is pseudo-Lipschitz in  $\theta$ . When  $f(x; \theta)$  is a differentiable convex function in  $x$  for every  $\theta$ , and  $h(x; \theta)$  is an affine function in  $x$  for every  $\theta$ , then  $X^*(\lambda; \theta)$  is the solution set of  $\text{VI}(X, \nabla_x \mathcal{L}_0(\cdot, \lambda; \theta))$  when  $\lambda \in \mathbb{R}_+^m$  and  $\theta \in \Theta$ . We consider two sets of problem classes in providing conditions under which the associated solution sets admit pseudo-Lipschitzian properties: 1) *Parametrized quadratic programming:* If  $f(x; \theta)$  is a quadratic function for every  $\theta \in \Theta$  and  $X$  is a polyhedral set, the mapping of the variational inequality problem is affine; such a problem is generally referred to as an affine variational inequality problem and denoted by  $\text{AVI}(X, M(\theta), q(\theta))$  where  $M(\theta)x + q(\theta) = \nabla_x f(x; \theta) + A(\theta)^\top \lambda$  while its solution set is denoted by  $\text{SOL}(X, M(\theta), q(\theta))$ ,  $\text{int}(K^+)$  denotes the interior of the positive dual cone of  $K$ , and  $K^+ \triangleq \{y : y^\top z \geq 0, \forall z \in K\}$ . Then under Theorem 7.4 [15], if  $M(\theta)$  is positive semidefinite over  $X$  for all  $\theta \in \Theta$ , and if  $q(\theta) \in \text{int}([\text{SOL}(X, M(\theta), 0)]^+)$ , then there exists scalars  $\varepsilon$  and  $\kappa$  such that if  $\max_{\hat{\theta} \in \Theta} \{\|M(\hat{\theta}) - M(\theta)\|, \|q(\hat{\theta}) - q(\theta)\|\} < \varepsilon$ , then

$$X^*(\lambda; \hat{\theta}) \subseteq X(\lambda; \theta) + \kappa \left( \|M(\hat{\theta}) - M(\theta)\| + \|q(\hat{\theta}) - q(\theta)\| \right) B(0, 1). \quad \text{Under a}$$

compactness assumption on  $\Theta$ , this “local” Lipschitzian result can be globalized. 2) *Parametrized convex programming*: More generally, suppose  $f(x; \theta)$  is a nonlinear convex function and  $B(H; \varepsilon, S)$  denotes an  $\varepsilon$ -neighborhood of  $H$  containing all continuous functions  $G$  that are within  $\varepsilon$  distance to  $H$  when restricted to the set  $S$ , i.e.,

$$\|G - H\|_S \triangleq \sup_{y \in S} \|G(y) - H(y)\| < \varepsilon.$$

Then we define the associated  $\text{VI}(X, \nabla_x \mathcal{L}(\cdot, \lambda; \theta))$  as *semi-stable* if there exist two positive scalars  $c$  and  $\varepsilon$  such that for every  $\nabla_x \mathcal{L}(\cdot, \lambda; \hat{\theta}) \in B(\nabla_x \mathcal{L}(\cdot, \lambda; \theta); \varepsilon, X)$ , we have that

$$\begin{aligned} X^*(\lambda; \hat{\theta}) &\subseteq X^*(\lambda; \theta) \\ &+ c \sup_{x \in X} \|\nabla_x \mathcal{L}(x, \lambda; \hat{\theta}) - \nabla_x \mathcal{L}(x, \lambda; \theta)\| B(0, 1). \end{aligned}$$

In fact, a necessary and sufficient condition for semi-stability of  $\text{VI}(X, F)$  is the following [16, Prop. 5.5.5]: There exists two positive scalars  $c$  and  $\varepsilon$ , such that for all  $q \in \mathbb{R}^n$ ,

$$\|q\| < \varepsilon \implies \text{SOL}(X, q + F) \subseteq \text{SOL}(X, F) + B(0, c\|q\|).$$

We conclude this section by presenting the misspecified variant of the inexact augmented Lagrangian scheme with constant penalty  $\rho > 0$ . Notably, if  $\theta_k = \theta^*$  for all  $k \geq 0$ , this reduces to the traditional version considered in [10].

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**Algorithm 1** Misspecified inexact aug. Lag. scheme

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Given  $\lambda_0 = \mathbf{0} \in \mathbb{R}^m$ , and  $\rho > 0$ , let  $\{\alpha_k\}, \{\theta_k\}$  be given sequences. Then for all  $k \geq 0$ ,

- (1) find  $x_k$  such that  $\mathcal{L}_\rho(x_k, \lambda_k; \theta_k) \leq g_\rho(\lambda_k; \theta_k) + \alpha_k$ ;
  - (2)  $\lambda_{k+1} = \lambda_k + \rho \nabla_\lambda \mathcal{L}_\rho(x_k, \lambda_k; \theta_k)$ ;
  - (3)  $k := k + 1$ ;
- 

*Assumption 3*:  $\{\alpha_k\}$  is chosen such that  $\sum_{k=0}^{\infty} \sqrt{\alpha_k} < \infty$ .

Under this assumption, we show (i)  $f^* - g_\rho(\bar{\lambda}_k) \leq \mathcal{O}(1/k)$  for  $\bar{\lambda}_k \triangleq \frac{1}{k} \sum_{i=1}^k \lambda_i$ , (ii)  $d_{\mathbb{R}^m}(h(\bar{x}_k; \theta^*)) \leq \mathcal{O}(1/\sqrt{k})$ , and (iii)  $-\mathcal{O}(1/\sqrt{k}) \leq f(\bar{x}_k; \theta^*) - f^* \leq \mathcal{O}(1/k)$  for  $\bar{x}_k \triangleq \frac{1}{k+1} \sum_{i=0}^k x_i$ . After proving these bounds independently, we became aware of related recent work [13], where Algorithm 1 is considered with  $\alpha_k = \alpha > 0$  for all  $k \geq 0$ , assuming *perfect information*, i.e.,  $\theta_k = \theta^*$  for all  $k \geq 0$ . In [13], it is shown that (i)  $f^* - g_\rho(\bar{\lambda}_k) \leq \mathcal{O}(1/k) + \alpha$ , (ii)  $d_{\mathbb{R}^m}(h(\bar{x}_k; \theta^*)) \leq \mathcal{O}(1/\sqrt{k})$ , and (iii)  $-\mathcal{O}(1/\sqrt{k}) \leq f(\bar{x}_k; \theta^*) - f^* \leq \mathcal{O}(1/k) + \alpha$ . Therefore, according to [13],  $\alpha$  should be fixed as a small constant in accordance with the desired accuracy. Since  $\alpha$  is fixed in [13], such avenues can, at best, provide approximate solutions. In contrast, our method may start with large  $\alpha_0$  and gradually decrease it, ensuring *both* numerical stability and asymptotic convergence to optimality.

### III. RATE OF CONVERGENCE ANALYSIS

We begin by showing that dual variables stay bounded by using a supporting Lemma whose proof follows from Lemma 1(i) and the properties of proximal maps (cf. [17]).

*Lemma 3*: Let  $\pi_\rho(\lambda; \theta) := \arg\max_{w \in \mathbb{R}^m} g_0(w; \theta) - \frac{1}{2\rho} \|w - \lambda\|^2$  for  $\theta \in \Theta$ , i.e., the proximal map of  $g_0(\cdot; \theta)$ . Then  $\pi_\rho(\lambda; \theta) = \lambda + \rho \nabla_\lambda g_\rho(\lambda; \theta)$ , and  $\pi_\rho$  is *nonexpansive* in  $\lambda$  for all  $\theta \in \Theta$ .

**Theorem 1 (Boundedness of  $\{\lambda_k\}$ )**: Let Assumptions 1–3 hold, and  $\lambda^*$  be an arbitrary solution to the Lagrangian dual of  $\mathcal{C}(\theta^*)$ , i.e.,  $\lambda^* \in \arg\max_\lambda g_0(\lambda; \theta^*)$ . Then for all  $k \geq 1$ ,  $\|\lambda_k - \lambda^*\| \leq C_\lambda$ , where  $C_\lambda$  is defined as follows:

$$C_\lambda \triangleq \sqrt{2\rho} \sum_{i=0}^{\infty} \sqrt{\alpha_i} + \rho M_h \frac{\|\theta_0 - \theta^*\|}{1-q} + \|\lambda^*\|. \quad (3)$$

*Proof*: We begin by deriving a bound on  $\|\lambda_{k+1} - \pi_\rho(\lambda_k; \theta_k)\|$  by utilizing the definition of  $\lambda_{k+1}$  from Step 2 in Algorithm 1:

$$\begin{aligned} &\|\lambda_{k+1} - \pi_\rho(\lambda_k; \theta_k)\| \\ &= \|\lambda_k + \rho \nabla_\lambda \mathcal{L}_\rho(x_k, \lambda_k; \theta_k) - \lambda_k - \rho \nabla_\lambda g_\rho(\lambda_k; \theta_k)\| \\ &= \rho \|\nabla_\lambda \mathcal{L}_\rho(x_k, \lambda_k; \theta_k) - \nabla_\lambda g_\rho(\lambda_k; \theta_k)\| \leq \sqrt{2\rho \alpha_k}, \end{aligned} \quad (4)$$

where the last inequality follows from Lemma 1 (iii). Since  $g_\rho(\cdot; \theta^*)$  is the Moreau regularization of  $g_0(\cdot; \theta^*)$ , it is true that  $\lambda^* \in \arg\max_\lambda g_\rho(\lambda, \theta^*)$  for all  $\rho > 0$ . Hence,  $\nabla_\lambda g_\rho(\lambda^*; \theta^*) = 0$  and  $\lambda^* = \pi_\rho(\lambda^*, \theta^*)$ . From this observation, we obtain the bound below:

$$\begin{aligned} \|\pi_\rho(\lambda_k, \theta_k) - \lambda^*\| &= \|\pi_\rho(\lambda_k, \theta_k) - \pi_\rho(\lambda^*, \theta^*)\| \\ &\leq \|\pi_\rho(\lambda_k, \theta_k) - \pi_\rho(\lambda_k, \theta^*)\| + \|\pi_\rho(\lambda_k, \theta^*) - \pi_\rho(\lambda^*, \theta^*)\| \\ &= \rho \|\nabla_\lambda g_\rho(\lambda_k, \theta_k) - \nabla_\lambda g_\rho(\lambda_k, \theta^*)\| \\ &\quad + \|\pi_\rho(\lambda_k, \theta^*) - \pi_\rho(\lambda^*, \theta^*)\| \\ &\leq \rho M_h \|\theta_k - \theta^*\| + \|\lambda_k - \lambda^*\|. \end{aligned} \quad (5)$$

This follows from the Lipschitz continuity of  $\nabla_\lambda g_\rho$  and the nonexpansivity of  $\pi_\rho$  in  $\lambda$  (Lemma 3). Hence, from (4) and (5), we obtain for all  $i \geq 0$  that

$$\|\lambda_{i+1} - \lambda^*\| \leq \sqrt{2\rho \alpha_i} + \rho M_h \|\theta_i - \theta^*\| + \|\lambda_i - \lambda^*\|.$$

For  $k \geq 0$ , by summing the above inequality over  $i = 0, \dots, k$ , and using the fact that  $\lambda_0 = \mathbf{0}$ , we get

$$\begin{aligned} \|\lambda_{k+1} - \lambda^*\| &\leq \sum_{i=0}^k \left( \sqrt{2\rho \alpha_i} + \rho M_h \|\theta_i - \theta^*\| \right) + \|\lambda_0 - \lambda^*\| \\ &\leq \sqrt{2\rho} \sum_{i=0}^{\infty} \sqrt{\alpha_i} + \rho M_h \frac{\|\theta_0 - \theta^*\|}{1-q} + \|\lambda^*\|. \end{aligned}$$

**Remark**: It is worth emphasizing that the bound  $C_\lambda$  can be tightened when  $\theta^*$  is known, i.e., since  $\theta_0 = \theta^*$ , the second term disappears. ■

Next, we prove that the augmented Lagrangian scheme generates a sequence  $\{\lambda_k\}$  such that  $\bar{\lambda}_k \rightarrow \lambda^*$  as  $k \rightarrow \infty$  by deriving a rate statement on the ergodic average sequence.

**Theorem 2 (Bound on dual suboptimality)**: Let Assumptions 1 – 3 hold and let  $\{\lambda_k\}_{k \geq 1}$  denote the sequence generated by Algorithm 1. In addition, let  $\bar{\lambda}_k \triangleq \frac{1}{k} \sum_{i=1}^k \lambda_i$ . Then it follows that for all  $k \geq 1$ :

$$f^* - g_\rho(\bar{\lambda}_k; \theta^*) = \sup_{\lambda} g_\rho(\lambda; \theta^*) - g_\rho(\bar{\lambda}_k; \theta^*) \leq \frac{B_g}{k}, \quad (6)$$

where  $\lambda^* \in \arg\max_\lambda g_0(\lambda, \theta^*)$ ,  $C_\lambda$  is defined in Theorem 1, and  $B_g$  is defined as follows:

$$B_g \triangleq \frac{1}{2\rho} \|\lambda^*\|^2 + C_\lambda \left( \sqrt{\frac{2}{\rho}} \sum_{k=0}^{\infty} \sqrt{\alpha_k} + \frac{M_h \|\theta_0 - \theta^*\|}{1-q} \right).$$

*Proof:* Note that from Lemma 1 and using the fact that the duality gap for  $\mathcal{C}(\theta^*)$  is 0, it follows that  $f^* = \max_{\lambda} g_{\rho}(\lambda; \theta^*)$  for all  $\rho > 0$ . Using the Lipschitz continuity of  $\nabla_{\lambda} g_{\rho}(\lambda, \theta^*)$  in  $\lambda$  with constant  $1/\rho$ , for  $i \geq 0$ , we get

$$-g_{\rho}(\lambda_{i+1}; \theta^*) \leq -g_{\rho}(\lambda_i; \theta^*) - \nabla_{\lambda} g_{\rho}(\lambda_i; \theta^*)^{\top} (\lambda_{i+1} - \lambda_i) + \frac{1}{2\rho} \|\lambda_{i+1} - \lambda_i\|^2. \quad (7)$$

Under the concavity of  $g_{\rho}(\lambda; \theta^*)$  in  $\lambda$ , we have that

$$-g_{\rho}(\lambda^*; \theta^*) \geq -g_{\rho}(\lambda_i; \theta^*) - \nabla_{\lambda} g_{\rho}(\lambda_i; \theta^*)^{\top} (\lambda^* - \lambda_i).$$

By combining the above inequality and (7), we get

$$\begin{aligned} & -g_{\rho}(\lambda_{i+1}; \theta^*) \\ & \leq -g_{\rho}(\lambda^*; \theta^*) - \nabla_{\lambda} g_{\rho}(\lambda_i; \theta^*)^{\top} (\lambda_{i+1} - \lambda^*) + \frac{1}{2\rho} \|\lambda_{i+1} - \lambda_i\|^2 \\ & = -g_{\rho}(\lambda^*; \theta^*) - \nabla_{\lambda} \mathcal{L}_{\rho}(x_i, \lambda_i; \theta_i)^{\top} (\lambda_{i+1} - \lambda^*) \\ & \quad + \delta_i^{\top} (\lambda_{i+1} - \lambda^*) + s_i^{\top} (\lambda_{i+1} - \lambda^*) + \frac{1}{2\rho} \|\lambda_{i+1} - \lambda_i\|^2 \\ & \leq -g_{\rho}(\lambda^*; \theta^*) - \frac{1}{\rho} (\lambda_{i+1} - \lambda_i)^{\top} (\lambda_{i+1} - \lambda^*) + \frac{1}{2\rho} \|\lambda_{i+1} - \lambda_i\|^2 \\ & \quad + \|\delta_i\| \|\lambda_{i+1} - \lambda^*\| + \|s_i\| \|\lambda_{i+1} - \lambda^*\|, \end{aligned} \quad (8)$$

where  $\delta_i \triangleq \nabla_{\lambda} g_{\rho}(\lambda_i; \theta_i) - \nabla_{\lambda} g_{\rho}(\lambda_i; \theta^*)$  and  $s_i \triangleq \nabla_{\lambda} \mathcal{L}_{\rho}(x_i, \lambda_i; \theta_i) - \nabla_{\lambda} g_{\rho}(\lambda_i; \theta_i)$ . By noting that  $\|\lambda_{i+1} - \lambda_i\|^2 + 2(\lambda_{i+1} - \lambda_i)^{\top} (\lambda^* - \lambda_{i+1}) = \|\lambda_i - \lambda^*\|^2 - \|\lambda_{i+1} - \lambda^*\|^2$ , we can rewrite (8) as

$$-g_{\rho}(\lambda_{i+1}; \theta^*) \leq -g_{\rho}(\lambda^*; \theta^*) + (\|\delta_i\| + \|s_i\|) \|\lambda_{i+1} - \lambda^*\| + \frac{1}{2\rho} (\|\lambda_i - \lambda^*\|^2 - \|\lambda_{i+1} - \lambda^*\|^2). \quad (9)$$

By summing (9) over  $i = 0, \dots, k-1$ , replacing  $g_{\rho}(\lambda^*; \theta^*)$  by  $f^* = \sup_{\lambda} g_{\rho}(\lambda, \theta^*)$  and setting  $\lambda_0 = \mathbf{0}$ , we obtain

$$\begin{aligned} & -\sum_{i=0}^{k-1} \left( g_{\rho}(\lambda_{i+1}; \theta^*) - f^* \right) + \frac{1}{2\rho} \|\lambda_k - \lambda^*\|^2 \\ & \leq \frac{1}{2\rho} \|\lambda^*\|^2 + \sum_{i=0}^{k-1} (\|\delta_i\| + \|s_i\|) \|\lambda_{i+1} - \lambda^*\|. \end{aligned} \quad (10)$$

Under concavity of  $g_{\rho}(\lambda; \theta^*)$  in  $\lambda$ , the following holds:

$$-\left( g_{\rho}(\bar{\lambda}_k; \theta^*) - f^* \right) \leq -\frac{1}{k} \sum_{i=0}^{k-1} \left( g_{\rho}(\lambda_{i+1}; \theta^*) - f^* \right).$$

By dividing both sides of (10) by  $k$  and dropping the positive term on the left hand side, we get

$$\begin{aligned} & f^* - g_{\rho}(\bar{\lambda}_k; \theta^*) \\ & \leq \frac{1}{k} \left( \frac{1}{2\rho} \|\lambda^*\|^2 + \sum_{i=0}^{k-1} (\|\delta_i\| + \|s_i\|) \|\lambda_{i+1} - \lambda^*\| \right). \end{aligned}$$

Lemma 1 and Lemma 2 imply that  $\|s_i\| \leq \sqrt{\frac{2\alpha_i}{\rho}}$ , and  $\|\delta_i\| \leq M_h \|\theta_i - \theta^*\|$ , resp., for all  $i \geq 0$ . In addition, from Theorem 1, we have  $\|\lambda_i - \lambda^*\| \leq C_{\lambda}$  for all  $i \geq 1$ . Then by the summability of  $\sqrt{\alpha_i}$ , we have that

$$\begin{aligned} & \sum_{i=0}^{\infty} (\|\delta_i\| + \|s_i\|) \|\lambda_{i+1} - \lambda^*\| \\ & \leq C_{\lambda} \left( M_h \sum_{i=0}^{\infty} \|\theta_i - \theta^*\| + \sqrt{\frac{2}{\rho}} \sum_{i=0}^{\infty} \sqrt{\alpha_i} \right). \end{aligned} \quad (11)$$

Furthermore, substituting  $\sum_{i=0}^{\infty} \|\theta_i - \theta^*\| = \|\theta_0 - \theta^*\|/(1-q)$  into (11) gives the desired bound and completes the proof.  $\blacksquare$

Next, we derive a bound on the primal *infeasibility*, where the primal iterate sequence is computed such that Step 1 in Algorithm 1 is satisfied. Prior to proving our main result, we provide some supporting technical lemmas.

**Lemma 4:** Assume that  $\phi(\lambda) : \mathbb{R}^m \rightarrow \mathbb{R}$  is a concave function whose supremum is finite and is attained at  $\lambda_{\phi}^*$ . In addition, assume that  $\nabla \phi$  is Lipschitz continuous with constant  $L_{\phi}$ . Then, for all  $\lambda \in \mathbb{R}^m$ , we have that  $\|\nabla \phi(\lambda)\| \leq \sqrt{2L_{\phi}(\phi(\lambda_{\phi}^*) - \phi(\lambda))}$ .

This is an immediate result of Theorem 2.1.5 in [18]. Next, we derive a bound on  $d_{\mathbb{R}_+^m}(y + y')$  for any  $y, y' \geq 0$ .

**Lemma 5:** For all  $y, y' \in \mathbb{R}_+^m$ ,  $d_{\mathbb{R}_+^m}(y + y') \leq d_{\mathbb{R}_+^m}(y) + \|y'\|$ .

*Proof:* The result immediately follows from the definition  $d_{\mathbb{R}_+^m}$  and the non-expansivity of  $\Pi_{\mathbb{R}_+^m}^c(x) \triangleq x - \Pi_{\mathbb{R}_+^m}(x)$ .  $\blacksquare$

We now derive the bound on the primal infeasibility.

**Theorem 3 (Bound on primal infeasibility):** Let

Assumptions 1–3 hold and let  $\{\lambda_k\}_{k \geq 0}$  and  $\{x_k\}_{k \geq 0}$  denote the sequences generated by Algorithm 1. Furthermore, let  $\bar{x}_k = \frac{1}{k+1} \sum_{i=0}^k x_i$ . Then, it follows that

$$d_{\mathbb{R}_+^m}(h(\bar{x}_k, \theta^*)) \leq \mathcal{V}(k) \triangleq \frac{C_1}{\sqrt{k+1}} + \frac{C_2}{k+1}, \quad (12)$$

where  $C_1 := \sqrt{\frac{2B_g}{\rho} + \left(\frac{C_{\lambda}}{\rho}\right)^2}$ , and  $C_2 := \sqrt{\frac{2}{\rho}} \sum_{i=0}^{\infty} \sqrt{\alpha_i} + \frac{(L_h + M_h) \|\theta_0 - \theta^*\|}{1-q}$ .

*Proof:* Let  $u_i := \nabla_{\lambda} \mathcal{L}_{\rho}(x_i, \lambda_i; \theta_i)$  for all  $i \geq 0$ . Note that computing  $\nabla_{\lambda} \mathcal{L}_{\rho}$  using (2), we get  $u_i = h(x_i; \theta_i) + \Pi_{\mathbb{R}_+^m} \left( -\frac{\lambda_i}{\rho} - h(x_i; \theta_i) \right)$ ; hence, it trivially follows that

$$h_j(x_i, \theta_i) \leq [u_i]_j, \text{ for } j = 1, \dots, m. \quad (13)$$

Under Assumption 1(iv), we have that

$$|h_j(x_i, \theta_i) - h_j(x_i, \theta^*)| \leq L_h^j \|\theta_i - \theta^*\|.$$

Combining this with (13), we obtain

$$h_j(x_i, \theta^*) \leq [u_i]_j + L_h^j \|\theta_i - \theta^*\|. \quad (14)$$

By summing (14) from  $i = 0$  to  $i = k$ , it follows that

$$\sum_{i=0}^k h_j(x_i, \theta^*) \leq \sum_{i=0}^k [u_i]_j + \sum_{i=0}^k L_h^j \|\theta_i - \theta^*\|. \quad (15)$$

On the other hand, convexity of  $h_j(x, \theta^*)$  in  $x$  implies that

$$h_j(\bar{x}_k, \theta^*) \leq \frac{1}{k+1} \sum_{i=0}^k h_j(x_i, \theta^*);$$

hence, for all  $j = 1, \dots, m$ , we have from (15),

$$h_j(\bar{x}_k, \theta^*) \leq \frac{1}{k+1} \left( \sum_{i=0}^k [u_i]_j + \sum_{i=0}^k L_h^j \|\theta_i - \theta^*\| \right). \quad (16)$$



Hence,  $L_h \triangleq \max\{L_h^j : j = 1, \dots, m\}$ , and (16) imply that

$$\begin{aligned} d_{\mathbb{R}^m} \left( h(\bar{x}_k, \theta^*) \right) &\leq \frac{1}{k+1} \left( \left\| \sum_{i=0}^k u_i \right\| + L_h \sum_{i=0}^k \|\theta_i - \theta^*\| \right) \\ &\leq \frac{1}{k+1} \left( \sum_{i=0}^k \|u_i\| + L_h \sum_{i=0}^k \|\theta_i - \theta^*\| \right). \end{aligned} \quad (17)$$

Recall that from Lemma 1 (iii), for  $i = 0, \dots, k$ ,

$$\left\| \nabla_{\lambda} \mathcal{L}_{\rho}(x_i, \lambda_i; \theta_i) - \nabla_{\lambda} g_{\rho}(\lambda_i; \theta_i) \right\| \leq \sqrt{\frac{2\alpha_i}{\rho}};$$

therefore, we obtain that  $\|u_i\| = \|\nabla_{\lambda} \mathcal{L}_{\rho}(x_i, \lambda_i; \theta_i)\| \leq \|\nabla_{\lambda} g_{\rho}(\lambda_i; \theta_i)\| + \sqrt{2\alpha_i/\rho}$ . In addition, since  $\|\nabla_{\lambda} g_{\rho}(\lambda_i; \theta_i)\| \leq \|\nabla_{\lambda} g_{\rho}(\lambda_i; \theta^*)\| + M_h \|\theta_i - \theta^*\|$ , we get the following bound:

$$\|u_i\| \leq \|\nabla_{\lambda} g_{\rho}(\lambda_i; \theta^*)\| + \sqrt{2\alpha_i/\rho} + M_h \|\theta_i - \theta^*\|.$$

On the other hand, by Lemma 4, we have

$$\|\nabla_{\lambda} g_{\rho}(\lambda_i; \theta^*)\| \leq \sqrt{\frac{2}{\rho} (f^* - g_{\rho}(\lambda_i; \theta^*))}.$$

Combining this with the previous inequality leads to

$$\|u_i\| \leq \sqrt{\frac{2}{\rho} (f^* - g_{\rho}(\lambda_i; \theta^*))} + \sqrt{\frac{2\alpha_i}{\rho}} + M_h \|\theta_i - \theta^*\|.$$

By substituting this bound into (17), we get that

$$\begin{aligned} d_{\mathbb{R}^m} \left( h(\bar{x}_k, \theta^*) \right) &\leq \frac{1}{k+1} \sum_{i=0}^k \sqrt{\frac{2}{\rho} (f^* - g_{\rho}(\lambda_i; \theta^*))} \\ &\quad + \frac{1}{k+1} \left( \sum_{i=0}^k \sqrt{\frac{2\alpha_i}{\rho}} + (L_h + M_h) \sum_{i=0}^k \|\theta_i - \theta^*\| \right) \\ &\leq \sqrt{\frac{2}{\rho} \left( f^* - \frac{1}{k+1} \sum_{i=0}^k g_{\rho}(\lambda_i; \theta^*) \right)} \\ &\quad + \frac{1}{k+1} \left( \sum_{i=0}^k \sqrt{\frac{2\alpha_i}{\rho}} + (L_h + M_h) \sum_{i=0}^k \|\theta_i - \theta^*\| \right), \end{aligned} \quad (18)$$

where the last inequality follows from concavity of square-root function  $\sqrt{\cdot}$ . The first term in (18) can be bounded using (10), which states that

$$\begin{aligned} f^* - \frac{1}{k+1} \sum_{i=0}^k g_{\rho}(\lambda_i; \theta^*) \\ \leq \frac{1}{k+1} (B_g + g_{\rho}(\lambda_0; \theta^*) - g_{\rho}(\lambda_{k+1}; \theta^*)). \end{aligned} \quad (19)$$

Note that  $g_{\rho}(\lambda_0; \theta^*) - f^* \leq 0$ , and using Lipschitz continuity of  $\nabla g_{\rho}$ , we have  $f^* - g_{\rho}(\lambda_{k+1}; \theta^*) \leq \frac{1}{2\rho} \|\lambda_{k+1} - \lambda^*\|^2 \leq \frac{1}{2\rho} C_{\lambda}^2$ . The remaining terms in (18) can also be bounded:

$$\begin{aligned} \frac{1}{k+1} \left( \sum_{i=0}^k \sqrt{\frac{2\alpha_i}{\rho}} + (L_h + M_h) \sum_{i=0}^k \|\theta_i - \theta^*\| \right) &\leq \\ \frac{1}{k+1} \left[ \sum_{i=0}^{\infty} \sqrt{\frac{2\alpha_i}{\rho}} + \frac{(L_h + M_h) \|\theta_0 - \theta^*\|}{1-q} \right]. \end{aligned} \quad (20)$$

The result follows by incorporating these bounds into (18).  $\blacksquare$

We now proceed to derive an upper bound on  $f(\bar{x}_k, \theta^*) - f^*$ . In contrast with standard unconstrained convex optimization,  $f(\bar{x}_k, \theta^*)$  could be less than  $f^*$ , as a consequence of infeasibility of  $\bar{x}_k$ .

**Theorem 4 (bound on primal suboptimality):** Let Assumption 1–3 hold and let  $\{\bar{x}_k\}$  and  $\{\lambda_k\}$  be the sequences generated by Algorithm 1. In addition, let  $\bar{x}_k = \frac{1}{k+1} \sum_{i=0}^k x_i$ . Then the following holds:

$$f(\bar{x}_k; \theta^*) - f^* \geq -\frac{\rho}{2} \mathcal{V}^2(k) - \|\lambda^*\| \mathcal{V}(k) \quad (21)$$

$$f(\bar{x}_k; \theta^*) - f^* \leq \frac{U}{k}, \quad (22)$$

for any  $\lambda^* \in \operatorname{argmax}_{\lambda} g_0(\lambda, \theta^*)$ , where  $\mathcal{V}(k)$  is defined in Theorem 3,  $U := \frac{\rho}{2} L_h^2 \frac{\|\theta_0 - \theta^*\|^2}{1-q^2} + (\bar{C}L_h + 2L_f) \frac{\|\theta_0 - \theta^*\|}{1-q} + \sum_{i=0}^{\infty} \alpha_i$ .

**Proof:** **Proof of the lower bound.** Since  $\sup_{\lambda} g_{\rho}(\lambda; \theta^*) = \min_{x \in X} \mathcal{L}_{\rho}(x, \lambda^*; \theta^*) = f^*$ , we have that for all  $k \geq 0$ ,

$$\begin{aligned} f^* &\leq \mathcal{L}_{\rho}(\bar{x}_k, \lambda^*; \theta^*) \\ &= f(\bar{x}_k; \theta^*) + \frac{\rho}{2} d_{\mathbb{R}^m}^2 \left( h(\bar{x}_k; \theta^*) + \frac{\lambda^*}{\rho} \right) - \frac{\|\lambda^*\|^2}{2\rho} \\ &\leq f(\bar{x}_k; \theta^*) + \frac{\rho}{2} \left( d_{\mathbb{R}^m} (h(\bar{x}_k; \theta^*)) + \frac{\|\lambda^*\|}{\rho} \right)^2 - \frac{\|\lambda^*\|^2}{2\rho}, \end{aligned}$$

where the first equality is a consequence of (2) while the second inequality follows from Lemma 5. By expanding the second term above inequality, we obtain

$$\begin{aligned} f^* &\leq f(\bar{x}_k; \theta^*) + \frac{\rho}{2} d_{\mathbb{R}^m}^2 (h(\bar{x}_k; \theta^*)) + d_{\mathbb{R}^m} (h(\bar{x}_k; \theta^*)) \|\lambda^*\| \\ &\leq f(\bar{x}_k; \theta^*) + \frac{\rho}{2} \mathcal{V}^2(k) + \|\lambda^*\| \mathcal{V}(k), \end{aligned}$$

where the last inequality follows from Theorem 3.

**Proof of the upper bound.** Let  $x^*$  be an optimal solution to  $\mathcal{C}(\theta^*)$ . Step 1 of Algorithm 1 implies that for all  $i \geq 0$

$$\mathcal{L}_{\rho}(x_i, \lambda_i; \theta_i) \leq \mathcal{L}_{\rho}(x^*, \lambda_i; \theta_i) + \alpha_i.$$

Hence, by the definition of  $\mathcal{L}_{\rho}$ , it follows that

$$\begin{aligned} f(x_i; \theta_i) + \frac{\rho}{2} d_{\mathbb{R}^m}^2 \left( h(x_i; \theta_i) + \frac{\lambda_i}{\rho} \right) - \frac{\|\lambda_i\|^2}{2\rho} &\leq \\ f(x^*; \theta_i) + \frac{\rho}{2} d_{\mathbb{R}^m}^2 \left( h(x^*; \theta_i) + \frac{\lambda_i}{\rho} \right) - \frac{\|\lambda_i\|^2}{2\rho} &+ \alpha_i, \end{aligned}$$

which leads to

$$\begin{aligned} f(x_i; \theta_i) - f(x^*; \theta_i) &\leq \\ \frac{\rho}{2} d_{\mathbb{R}^m}^2 \left( h(x^*; \theta_i) + \frac{\lambda_i}{\rho} \right) - \frac{\rho}{2} d_{\mathbb{R}^m}^2 \left( h(x_i; \theta_i) + \frac{\lambda_i}{\rho} \right) &+ \alpha_i. \end{aligned} \quad (23)$$

Step 2 of Algorithm 1 implies that

$$d_{\mathbb{R}^m} \left( h(x_i; \theta_i) + \frac{\lambda_i}{\rho} \right) = \frac{\|\lambda_{i+1}\|}{\rho}. \quad (24)$$

In addition, by using Lemma 5, it follows that

$$d_{\mathbb{R}^m} \left( h(x^*; \theta_i) + \frac{\lambda_i}{\rho} \right) \leq d_{\mathbb{R}^m} (h(x^*; \theta_i)) + \frac{\|\lambda_i\|}{\rho}. \quad (25)$$

Substituting (24) and (25) in (23), we obtain for all  $i \geq 0$

$$\begin{aligned} f(x_i; \theta_i) - f(x^*; \theta_i) &\leq \frac{\rho}{2} \left( d_{\mathbb{R}^m} (h(x^*; \theta_i)) + \frac{\|\lambda_i\|}{\rho} \right)^2 \\ &\quad - \frac{1}{2\rho} \|\lambda_{i+1}\|^2 + \alpha_i = \frac{\rho}{2} d_{\mathbb{R}^m}^2 (h(x^*; \theta_i)) + d_{\mathbb{R}^m} (h(x^*; \theta_i)) \|\lambda_i\| \\ &\quad + \frac{1}{2\rho} (\|\lambda_i\|^2 - \|\lambda_{i+1}\|^2) + \alpha_i. \end{aligned} \quad (26)$$

From Lipschitz continuity of  $h_j$  in  $\theta$  for  $j = 1, \dots, m$ ,

$$\begin{aligned} h_j(x^*; \theta_i) &\leq h_j(x^*; \theta^*) + L_h \|\theta_i - \theta^*\|; \\ \implies d_{\mathbb{R}^m} (h(x^*; \theta_i)) &\leq d_{\mathbb{R}^m} (h(x^*; \theta^*)) + L_h \|\theta_i - \theta^*\|. \end{aligned} \quad (27)$$

Since  $h_j(x^*; \theta^*) \leq 0$  for  $j = 1, \dots, m$ , it follows that  $d_{\mathbb{R}^m} (h(x^*; \theta^*)) = 0$ , and inequality (27) becomes

$$d_{\mathbb{R}^m} (h(x^*; \theta_i)) \leq L_h \|\theta_i - \theta^*\|.$$

By substituting (27) into (26), we get for all  $i \geq 0$

$$\begin{aligned} f(x_i; \theta_i) - f(x^*; \theta_i) &\leq \frac{\rho}{2} L_h^2 \|\theta_i - \theta^*\|^2 + \bar{C} L_h \|\theta_i - \theta^*\| \\ &\quad + \frac{1}{2\rho} (\|\lambda_i\|^2 - \|\lambda_{i+1}\|^2) + \alpha_i, \end{aligned}$$

where the last inequality follows from  $\|\lambda_i - \lambda^*\| \leq C_\lambda$  (Theorem 1), i.e.,  $\|\lambda_i\| \leq \bar{C} := C_\lambda + \|\lambda^*\|$  for all  $i \geq 0$ . Next, from the Lipschitz continuity of  $f$  in  $\theta$ , it follows that

$$f(x_i; \theta_i) - f(x^*; \theta_i) \geq f(x_i; \theta^*) - f(x^*; \theta^*) - 2L_f \|\theta_i - \theta^*\|.$$

Combining two above inequalities results in the following:

$$\begin{aligned} f(x_i; \theta^*) - f^* &\leq \frac{\rho}{2} L_h^2 \|\theta_i - \theta^*\|^2 + (\bar{C} L_h + 2L_f) \|\theta_i - \theta^*\| \\ &\quad + \frac{1}{2\rho} (\|\lambda_i\|^2 - \|\lambda_{i+1}\|^2) + \alpha_i. \end{aligned}$$

Summing the above inequality for  $i = 0$  to  $k$ , we get

$$\begin{aligned} \sum_{i=0}^k (f(x_i; \theta^*) - f^*) &\leq \frac{\rho}{2} L_h^2 \sum_{i=0}^k \|\theta_i - \theta^*\|^2 + \sum_{i=1}^k \alpha_i \\ &\quad + (\bar{C} L_h + 2L_f) \sum_{i=0}^k \|\theta_i - \theta^*\| + \frac{1}{2\rho} \sum_{i=0}^k (\|\lambda_i\|^2 - \|\lambda_{i+1}\|^2) \\ &\leq \frac{\rho}{2} L_h^2 \frac{\|\theta_0 - \theta^*\|^2}{1 - q^2} + (\bar{C} L_h + 2L_f) \frac{\|\theta_0 - \theta^*\|}{1 - q} + \sum_{i=0}^{\infty} \alpha_i. \end{aligned}$$

Since  $f(x; \theta)$  is convex in  $x$ , dividing both sides of the above inequality by  $k$  gives the desired result. ■

#### IV. CONCLUSION

In this paper, we consider the setting of an optimization problem complicated by misspecification both in the function and in the constraints. The parameter misspecification may be resolved through a learning problem. Suppose we have access to a learning data set, collected a priori. One avenue for contending with such a problem is through an inherently sequential approach that solves the learning problem and utilizes this solution in subsequently solving the computational problem. Unfortunately, unless accurate solutions of

the learning problem are available in a finite number of iterations, sequential approaches can only provide approximate solutions. Instead, we focus on a simultaneous approach that combines learning and computation by adopting inexact augmented Lagrangian (AL) scheme with constant penalty parameter. In this regard, we made the following contributions: (i) Derivation of the convergence rate for dual optimality, primal infeasibility and primal suboptimality; (ii) Quantification of the effect of learning on the rate.

Our future work lies in deriving the overall iteration complexity analysis, which incorporates the number of iterations required to solve the subproblems arising in the AL scheme and quantify the resulting impact of learning.

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